Light-front gauge invariant formulation and electromagnetic duality

2005 J. Phys. A: Math. Gen. 38 7021

(http://iopscience.iop.org/0305-4470/38/31/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 7021–7035 doi:10.1088/0305-4470/38/31/011

Light-front gauge invariant formulation and electromagnetic duality

J A Przeszowski

Institute of Theoretical Physics, University of Białystok, ul. Lipowa 41, 15-424 Białystok, Poland

E-mail: jprzeszo@alpha.uwb.edu.pl

Received 11 May 2005
Published 20 July 2005
Online at stacks.iop.org/JPhysA/38/7021

Abstract
The gauge invariant formulation of Maxwell's equations and the electromagnetic duality transformations are given in the light-front (LF) variables. The novel formulation of the LF canonical quantization, which is based on the kinematic translation generator P^+ rather than on the Hamiltonian P^-, is proposed. This canonical quantization is applied for the free electromagnetic fields and for the fields generated by electric and magnetic external currents. The covariant form of photon propagators, which agrees with Schwinger's source theory, is achieved when the direct interaction of external currents is properly chosen. Applying the path-integral formalism, the equivalent LF Lagrangian density, which depends on two Abelian gauge potentials, is proposed. Some remarks on the Dirac strings and LF non-local structures are presented in the appendix.

PACS numbers: 11.10.Ef, 11.15.Bt, 11.30.Er, 12.20.Ds

1. Introduction

The light-front (LF) formulation of Maxwell's electromagnetic theory with both the electric and magnetic external currents is interesting for at least two reasons: first as a relativistic field theoretical model and second as an introduction to the dual description of QCD. The crucial step for such a formulation is the electromagnetic duality transformation, which allows one to add magnetic currents into the electromagnetic system with electric currents.

The first attempt in this direction, restricted to a classical theory, done by Gambini and Salamó [1], is partially successful. They have selected two gauge invariant independent degrees of freedom and they have formulated the continuous LF electromagnetic duality transformation in terms of these degrees of freedom. They have shown that this duality transformation is consistent with the usual duality transformation for electric and magnetic fields $\vec{E}$ and $\vec{B}$. However, when trying to incorporate the Cabibbo–Ferrari theory [2] into the

LF formulation, they found incorrectly four gauge invariant independent degrees of freedom. Also Gambini and Salamó have proposed effective Lagrangians with noncanonical current terms.

Next, some five to six years ago, there were two independent attempts to describe the discrete electromagnetic duality within the LF description of Maxwell's theory. They were initiated by Susskind [3], who has argued that the LF electromagnetic duality transformation can be expressed simply in terms of the transverse potentials as

$$A_i \longmapsto -\epsilon_{ij} A_j. \tag{1}$$

This conjecture has been explicitly checked by Brisudova [4], within the LF canonical quantization in the LC gauge $A_- = A^+ = 0$, with a final conclusion that only for free fields can one define such duality. This follows directly from the starting point, since any one gauge potential description of electromagnetic theory is evidently false for a theory with electric and magnetic currents. Instead one should use either a two gauge potential approach or a LF version of Dirac strings. Also the idea of entanglement of duality with gauge transformations [3, 4] is physically misleading—these transformations are fundamentally different and should be treated separately.

The next attempt, by Mukherjee and Bhattacharya [5], is based on a different approach with two gauge potentials. This method, introduced long ago by Zwanzinger [6] within the equal-time (ET) formulation, leads to quite a complicated picture, where one has to introduce a constant space-like vector $n^\mu, n^2 < 0$, which breaks the Lorentz invariance of starting Lagrangian density. This Lagrangian leads to many constraints, which follow from two gauge symmetries. Mukherjee and Bhattacharya, implementing the Dirac canonical quantization procedure for systems with constraints [7], found that effectively only one component of each gauge potential is an independent degree of freedom. Therefore, after redefining these independent modes as two transverse components of some effective gauge potential they prove the Susskind conjecture (1) for the interacting electromagnetic theory with both electric and magnetic external currents. Their analysis of the quantum theory ends with the structure of LF commutators and Hamiltonian. Since their Hamiltonian contains a rotationally noninvariant term, which describes instantaneous interaction of electric and magnetic currents, then we may worry if a covariant perturbation theory follows from their analysis.

The aim of our paper is to formulate a clear LF description of Maxwell's theory, where only the gauge invariant objects are used. We will start with the free LF Maxwell equations and define the LF electromagnetic duality transformation as their symmetry. Then we will quantize the system canonically, treating x^+ as the LF evolution parameter and using $\bar{x} = (x^-, x^2, x^3)$ as coordinates on the LF (hyper-)surface. The Poincaré generators will be defined in terms of the symmetric energy–momentum tensor $T^{\mu\nu}_{\text{sym}}$, which is gauge invariant, rather than the canonical energy–momentum tensor $T^{\mu\nu}_{\text{can}}$, which is a gauge dependent object. We will propose a novel canonical procedure for the LF systems, where the generator of translations in the x^- direction is used for deriving canonical LF Poisson brackets for all independent fields. Since this generator is a kinematic one, it will keep its free field form also for an interacting theory, where a Hamiltonian contains some interaction part. Our canonical procedure is explicitly duality invariant and we will prove the Susskind conjecture (1), but for fields with a different physical interpretation.

Our paper is organized as follows. In section 2 we present a very concise ET description of Maxwell's equations with external electric and magnetic sources. In section 3 we start with the LF tensor formulation of Maxwell's equations and the electromagnetic duality transformation. We also introduce the LF notation for the electromagnetic fields. In section 4 we analyse the free field case, when no external sources are present. In section 5 we consider the general case of the Maxwell equations, when both kinds of external sources are present, paying special attention to the LF duality transformation. In section 6 we switch our analysis to the path-integral formulation and derive all propagators, which mediate interactions between electric and magnetic currents. We also find a local Lagrangian, which is equivalent to the proposed Hamiltonian formulation. In conclusions we discuss our results indicating the crucial points and suggesting possible further investigations. The LF notation and the Green functions are presented in appendix A. In appendix B we give a short presentation of different Dirac's strings for the ET and LF formulations.

2. Vector notation for Maxwell's equations

Before starting our novel LF approach, let us briefly review the ET formulation. The magnetic charges ρ_m and currents $\vec{J}_m$ are formally added to the electric charges ρ_e and currents $\vec{J}_e$, when one writes Maxwell's equations

$$\vec{\nabla} \times \vec{B} = \frac{\partial}{\partial t}\vec{E} + \vec{J}_e, \qquad \vec{\nabla} \cdot \vec{E} = \rho_e, \tag{2a}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial}{\partial t}\vec{B} - \vec{J}_m, \qquad \vec{\nabla} \cdot \vec{B} = \rho_m. \tag{2b}$$

This set of equations is invariant under the electromagnetic duality transformation

$$\vec{E} \longmapsto \vec{B} \longmapsto -\vec{E}, \tag{3a}$$

$$\vec{J}_e \longmapsto \vec{J}_m \longmapsto -\vec{J}_e, \tag{3b}$$

$$\rho_e \longmapsto \rho_m \longmapsto -\rho_e. \tag{3c}$$

This vector notation is not a suitable starting point for the LF formulation, since for 3-vectors, in contrast to 4-vectors, there is no definition of LF components.

Usually, when there are no magnetic sources ($\rho_m = \vec{J}_m = 0$), then a pair of homogeneous Maxwell's equations (2b), may be removed with the help of 4-vectors $A_\mu = (A_0, \vec{A})$ by the standard definitions of gauge field potentials:

$$\vec{E} = -\vec{\nabla} A_0 - \frac{\partial}{\partial t}\vec{A}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}. \tag{4}$$

This allows us to express the inhomogeneous Maxwell equations (2a) in terms of 4-vectors A_μ, which further can be easily converted into the LF notation. This may explain why, in almost all papers on the LF quantization of Maxwell's theory, only the gauge field potential approach has been used (for reviews see [9, 10]).

However if magnetic sources are present, then the formulation in terms of a single gauge field potential runs into inconsistencies. The way out is either by means of the Dirac string [11, 12] or by the Wu–Yang potentials [13], to mention only the best known solutions. But we would like to stress that the gauge potentials are only quite useful, but are not necessary, for a consistent quantization of Maxwell's equations (2a), (2b) within the ET approach [14, 15]. Actually, they couple locally to external currents, but the canonical commutation relations can be solely expressed in terms of the electromagnetic fields $(\vec{E}, \vec{B})$. Thus we expect that also within the LF approach one can consistently quantize Maxwell's equations directly in terms of electromagnetic fields.

3. LF notation for Maxwell's equations and electromagnetic duality

As a starting point for the LF formulation of Maxwell's electromagnetism with classical external electric sources, we take the tensor formulation: $\partial_\mu F^{\mu\nu} = J^\mu, \epsilon^{\mu\nu\lambda\rho}\partial_\nu F_{\lambda\rho} = 0$. These equations can be easily transformed into the LF coordinates as the inhomogeneous LF Maxwell equations:

$$\partial_+ E_- = \partial_i E_i + J^-, \tag{5a}$$

$$\partial_+ B_i = -\partial_- E_i - \epsilon_{ij}\partial_j B_- + J^i, \tag{5b}$$

$$0 = \partial_- E_- + \partial_i B_i + J^+, \tag{5c}$$

and the LF Bianchi identities:

$$\partial_+ B_i = \partial_- E_i - \partial_i E_-, \tag{6a}$$

$$\partial_+ B_- = \epsilon_{ij}\partial_i E_j, \tag{6b}$$

$$\partial_- B_- = \epsilon_{ij}\partial_i B_j, \tag{6c}$$

where the LF electromagnetic fields are defined as $E_- = F_{+-}, E_i = F_{+i}, B_i = F_{-i}, \epsilon_{ij}B_- = F_{ij}$. These equations, being manifestly gauge invariant, may be taken as the basis for the further canonical procedure, both for the free field case and the interacting field case.

Next, the tensor form of the electromagnetic duality transformations

$$F_{\mu\nu} \longmapsto \star F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\mu\nu\lambda\sigma}F^{\lambda\sigma}, \tag{7}$$

can be rewritten in the LF coordinates as

$$E_- \longmapsto B_-, \qquad B_- \longmapsto -E_-, \tag{8a}$$

$$E_i \longmapsto \epsilon_{ij}E_j, \qquad B_i \longmapsto -\epsilon_{ij}B_j, \tag{8b}$$

and hereafter we will refer to them as the LF electromagnetic duality transformation. If one doubts that (8b) is correct, then one may introduce another notation

$$\bar{E}_2 = E_2, \qquad \bar{E}_3 = -B_2, \qquad \bar{B}_2 = E_3, \qquad \bar{B}_3 = B_3, \tag{9a}$$

which allows us to express (8b) as

$$\bar{E}_i \longmapsto \bar{B}_i, \qquad \bar{B}_i \longmapsto -\bar{E}_i. \tag{9b}$$

However, we think that the very form of the LF electromagnetic duality is not important—what really matters is how these transformations act on Maxwell's equations. One can easily convince oneself that in the absence of external sources ($J^\mu = 0$) the LF Maxwell equations (5a)–(5c) and (6a)–(6c) transform, mutually in pairs, under (8a), (8b). Since the duality transformation for sources (3b), (3c) can be directly expressed in terms of the 4-currents $J^\mu = (\rho_e, \vec{J}_e)$ and $K^\mu = (\rho_m, \vec{J}_m)$

$$J^\mu \longmapsto K^\mu \longmapsto -J^\mu, \tag{10a}$$

then for the LF components one has

$$J^\pm \longmapsto K^\pm \longmapsto -J^\pm, \qquad J^i \longmapsto K^i \longmapsto -J^i. \tag{10b}$$

Finally, we also need a gauge invariant energy–momentum tensor, thus we take the symmetric energy–momentum tensor

$$T^{\mu\nu}_{\rm sym} = F^\mu_\lambda F^{\lambda\nu} + \tfrac{1}{4}g^{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}, \tag{11}$$

with the LF components

$$T^{+-}_{\rm sym} = \tfrac{1}{2}\left(E_-^2 + B_-^2\right), \qquad T^{++}_{\rm sym} = B_i^2, \qquad T^{+i}_{\rm sym} = E_- B_i - \epsilon_{ij} B_j B_-, \tag{12a}$$

$$T^{-i}_{\rm sym} = -E_- E_i - \epsilon_{ij} E_j B_-, \qquad T^{--}_{\rm sym} = E_i^2, \tag{12b}$$

$$T^{ij}_{\rm sym} = -(E_i B_j + E_j B_i) + \delta_{ij}\left(E_k B_k + \tfrac{1}{2}E_-^2 + \tfrac{1}{2}B_-^2\right). \tag{12c}$$

These components of $T^{\mu\nu}_{\rm sym}$ are invariant under the LF duality transformations (8*a*), (8*b*) and in our further investigations we will always keep our quantization procedure both gauge and duality invariant.

4. Gauge invariant canonical quantization for free fields

In this section we would like to focus our attention on the quantization of electromagnetic degrees of freedom, thus we will consider the case of free electromagnetic fields when all external sources vanish ($J^\mu = 0$). In our LF canonical procedure we choose x^+ as the temporal evolution parameter, thus all classical LF Maxwell equations (5*a*)–(6*c*) should be classified either as equations of motion or constraints. Usually, when an equation contains a term with the temporal partial derivative ∂_+ it is classified as an equation of motion, otherwise it is a constraint. However, here the situation is more tricky, since both equations (5*b*), (6*a*) contain terms $\partial_+ B_i$, thus their linear combination gives rise to the effective constraints

$$2\partial_- E_i = \partial_i E_- - \epsilon_{ij}\partial_j B_-, \tag{13}$$

and the effective equations of motion

$$2\partial_+ B_i = -\partial_i E_- - \epsilon_{ij}\partial_j B_-. \tag{14}$$

Thus we conclude that for Maxwell's theory there are more LF constraints than ET ones. A formal reason for this difference follows from the inhomogeneous equation (5*b*), which being an Euler–Lagrange equation, generally contains the temporal derivative term $\partial_+ D_i$, where the canonical momentum field is defined as

$$D_i = \frac{\partial \mathcal{L}}{\partial E_i}. \tag{15a}$$

In contrast, the homogeneous equation (6*b*), being a Bianchi identity, contains the term $\partial_+ B_i$. Generally, these two fields (D_i, B_i) form a canonical pair and are independent degrees of freedom. However, for Maxwell's theory, with the Lagrangian density

$$\mathcal{L} = -\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} = \tfrac{1}{2}\left(E_-^2 - B_-^2\right) + E_i B_i, \tag{15b}$$

one finds that these canonical variables (D_i, B_i) are constrained

$$D_i = B_i, \tag{15c}$$

and two different equations contain the same term $\partial_+ B_i$. Further consequences of this observation will be presented elsewhere and here we only stress that the constraint (15*c*) is gauge invariant, thus truly a physical phenomenon, which can be intimately connected with the Lorentz symmetry of Maxwell's electrodynamics.

Since there is no equation of motion for E_i, then we would like to treat these fields as dependent field variables and remove them from our canonical analysis by means of the constraint equations (13).[1]

[1] This is quite similar to the nondynamical components of fermion field ψ_-, $\psi_-^\dagger$, which are removed by solving the constraint part of Dirac's equations.

All remaining LF electromagnetic fields (E_-, B_-, B_i) have their equations of motion

$$(2\partial_+\partial_- - \triangle_\perp)E_- = 0, \tag{16a}$$

$$(2\partial_+\partial_- - \triangle_\perp)B_- = 0, \tag{16b}$$

$$2\partial_+ B_i = -\partial_i E_- - \epsilon_{ij}\partial_j B_-, \tag{16c}$$

and also appear in the constraints[2]

$$\partial_- E_- + \partial_i B_i = 0, \tag{17a}$$

$$\partial_- B_- - \epsilon_{ij}\partial_i B_j = 0. \tag{17b}$$

Further we note that only these LF fields appear in the gauge invariant Poincaré generators of translations

$$P^- = \int \mathrm{d}^3\bar{x}\, T^{+-}_{\mathrm{sym}} = \int \mathrm{d}^3\bar{x}\frac{1}{2}(E_-^2 + B_-^2), \tag{18a}$$

$$P^+ = \int \mathrm{d}^3\bar{x}\, T^{++}_{\mathrm{sym}} = \int \mathrm{d}^3\bar{x}\, B_i^2, \tag{18b}$$

$$P^i = \int \mathrm{d}^3\bar{x}\, T^{+i}_{\mathrm{sym}} = \int \mathrm{d}^3\bar{x}(E_- B_i - \epsilon_{ij} B_j B_-). \tag{18c}$$

Therefore, using the equations of motion (16*a*)–(16*c*), one can prove that these generators are the constants of motion (do not depend on x^+).

Since our system contains constraints for canonical field variables, then our next steps within the canonical quantization procedure could be taken according to either the Dirac method [7] or the Faddeev–Jackiw method of the Hamiltonian reduction [8]. However, we prefer to choose a less technical procedure and solve the constraint equations (17*a*), (17*b*) by a suitable parametrization of electromagnetic fields:

$$E_- = -\partial_i \mathcal{A}_i, \qquad B_- = \epsilon_{ij}\partial_i\mathcal{A}_j, \qquad B_i = \partial_-\mathcal{A}_i, \tag{19}$$

where $\mathcal{A}_i$ are the independent gauge invariant fields. This parametrization clearly shows that within the LF formulation, there are only two independent dynamical modes with the equations of motion

$$(2\partial_+\partial_- - \triangle_\perp)\mathcal{A}_i = 0. \tag{20}$$

Thus we have complete agreement between the ET and LF formulations: in both approaches we have two independent relativistic modes, which are described by four canonical ET fields and two canonical LF fields.

This observation agrees with [1], though Gambini and Salamó take E_- and B_- (using a different notation for them) as the independent electromagnetic modes. Also it forms the physical explanation of the additional constraints (13), which appear in the LF approach—there is only one independent canonical field variable for any relativistic independent mode, in contrast to the ET approach, where every mode is described by two canonical fields.

Using our parametrization of fields, we may express the gauge invariant Poincaré generators as

[2] This is quite analogous to the status of ET electromagnetic fields $(\vec{E}, \vec{B})$.

$$P^- = \int d^3\bar{x}\, T^{+-}_{\text{sym}} = \frac{1}{2}\int d^3\bar{x}[(\partial_i\mathcal{A}_i)^2 + (\epsilon_{ij}\partial_i\mathcal{A}_j)^2], \tag{21a}$$

$$P^+ = \int d^3\bar{x}\, T^{++}_{\text{sym}} = \int d^3\bar{x}(\partial_-\mathcal{A}_i)^2, \tag{21b}$$

$$P^i = \int d^3\bar{x}\, T^{+i}_{\text{sym}} = -\int d^3\bar{x}(\partial_i\mathcal{A}_j\partial_-\mathcal{A}_j), \tag{21c}$$

while the LF duality electromagnetic transformations (8a), (8b) boil down to

$$\mathcal{A}_i \mapsto -\epsilon_{ij}\mathcal{A}_j, \tag{22}$$

which agrees with Susskind's conjecture (1). Here we note that our results have the same form as those found within the usual gauge potential approach for the LC gauge condition $A_- = A^+ = 0$, provided we set $\mathcal{A}_i = A_i$. We stress that it is just a coincidence, since our variables $\mathcal{A}_i$, being gauge invariant quantities, are true physical fields and there is no gauge transformation for them. For free Maxwell's theory, when one chooses the LC gauge condition $A_- = A^+ = 0$, all nonphysical modes are removed and the transverse potentials A_i describe physical modes. However, for an interacting theory, ignoring the difference between $\mathcal{A}_i$ and A_i, one may run into serious difficulties—just like in [4].

Also we would like to indicate another difference between the gauge independent field variables $\mathcal{A}_i$ and the usual gauge field potential A_μ. In our case, we use $\mathcal{A}_i$ to solve identically two LF constraints (17a), (17b), while the usual vector gauge potentials A_μ solve identically all Bianchi identities (3), (6a)–(6c).

Since we have effectively reduced the constrained system (E_-, B_-, B_i) into the independent dynamical fields $\mathcal{A}_i$, then the LF canonical procedure may follow directly from the Poisson bracket relation

$$\partial_+\mathcal{A}_i = \{\mathcal{A}_i, P^-\}_{PB}, \tag{23}$$

when we take the equations of motion (20) and the LF Hamiltonian P^- (21a). However, here we would like to propose a novel LF canonical procedure, which uses P^+ instead of P^-. Thus we argue that one may start with the trivial relation

$$\partial_-\mathcal{A}_i = \{\mathcal{A}_i, P^+\}_{PB}, \tag{24}$$

and using expression (21b) for P^+, one may easily infer the canonical LF Poisson bracket

$$2\{\partial_-\mathcal{A}_i(x^+, \bar{x}), \mathcal{A}_j(x^+, \bar{y})\}_{PB} = -\delta_{ij}\delta^3(\bar{x} - \bar{y}). \tag{25}$$

Since P^+ is the kinematic generator, then relation (24) can be effectively used for finding the LF canonical brackets also for an interacting theory.

For a consistency check, one may use the LF canonical brackets (25) to calculate other Poisson brackets

$$\partial^\mu\mathcal{A}_i = \{\mathcal{A}_i, P^\mu\}_{PB}, \tag{26}$$

finding (for $\mu = -$) the proper form of the equations of motion (20) and (for $\mu = j$) the trivial identities.

The canonical quantization procedure means that canonical variables are changed into quantum operators and Poisson brackets are transformed into commutators[3]

$$2[\partial_-\mathcal{A}_i(x^+, \bar{x}), \mathcal{A}_j(x^+, \bar{y})] = -i\delta_{ij}\delta^3(\bar{x} - \bar{y}). \tag{27}$$

We stress that both the classical brackets and the quantum commutators are invariant under the LF duality transformation (22), thus our canonical quantization procedure is both gauge and duality invariant.

[3] We will denote the quantum field operators by the same symbols as the respective classical fields hoping that this will not lead to any misunderstanding.

5. Electric and magnetic external currents

As a next step we would like to consider the case of LF electromagnetic fields interacting with electric and magnetic currents. Therefore, we take the inhomogeneous Maxwell equations with electric external currents (5*a*)–(5*c*) and applying the duality transformations (8*a*), (8*b*), (10*b*) we generate the equations with magnetic currents

$$\partial_+ E_- = \partial_i E_i + J^-, \qquad \partial_+ B_- = \epsilon_{ij}\partial_i E_j + K^-, \tag{28a}$$

$$\partial_+ B_i = -\partial_- E_i - \epsilon_{ij}\partial_j B_- + J^i, \qquad \partial_+ B_i = \partial_- E_i - \partial_i E_- + \epsilon_{ij} K^j, \tag{28b}$$

$$\partial_- E_- = -\partial_i B_i - J^+, \qquad \partial_- B_- = \epsilon_{ij}\partial_i B_j - K^+. \tag{28c}$$

These LF Maxwell's equations with electric and magnetic currents are equivalent to (2*a*) and (2*b*) and they form the starting point for the analysis of interacting theory. There are two consistency conditions for these Maxwell's equations: the electric current continuity equation

$$\partial_+ J^+ + \partial_- J^- + \partial_i J^i = 0, \tag{29a}$$

and the magnetic current continuity equation

$$\partial_+ K^+ + \partial_- K^- + \partial_i K^i = 0. \tag{29b}$$

In this paper, we will not analyse the dynamical structure of these electric and magnetic currents, thus we will only suppose that these two conservation laws always hold. We stress that all equations (28*a*)–(28*c*) are inhomogeneous, thus we argue that generally no equation can be interpreted as a Bianchi identity and there is no one gauge potential approach which would lead to these equations.

We will carry out our LF quantization procedure following the same steps as in the previous section and start with the identification of equations of motion and constraints. As before, E_i is a nondynamical field variable, which now can be determined from the effective constraint

$$2\partial_- E_i = \partial_i E_- - \epsilon_{ij}\partial_j B_- + J^i - \epsilon_{ij} K^j. \tag{30}$$

The other fields are dynamical with the effective equations of motion

$$2\partial_+ B_i = -\epsilon_{ij}\partial_j B_- - \partial_i E_- + J^i + \epsilon_{ij} K^j, \tag{31a}$$

$$(2\partial_+\partial_- - \triangle_\perp) B_- = \epsilon_{ij}\partial_i J^j + \partial_i K^i + 2\partial_- K^-, \tag{31b}$$

$$(2\partial_+\partial_- - \triangle_\perp) E_- = -\epsilon_{ij}\partial_i K^j + \partial_i J^i + 2\partial_- J^-, \tag{31c}$$

while the remaining constraint equations are as follows:

$$\partial_- E_- + \partial_i B_i = -J^+, \tag{32a}$$

$$\partial_- B_- - \epsilon_{ij}\partial_i B_j = -K^+. \tag{32b}$$

These last equations can be interpreted as the LF electric Gauss law and the LF magnetic Gauss law, respectively. Similarly to the previous section, we wish to find a useful parametrization of the LF electromagnetic fields, which would solve identically the above constraints. However, now the best choice is no longer evident, since we have to allow for some explicit dependence of electromagnetic fields on the external LF charges J^+ and K^+. Therefore, we may try different possibilities, for example, we may take either

$$E_- = -\partial_i \mathcal{A}_i - (\partial_-)^{-1} J^+, \tag{33a}$$

$$B_- = \epsilon_{ij} \partial_i \mathcal{A}_j - (\partial_-)^{-1} K^+, \tag{33b}$$

$$B_i = \partial_- \mathcal{A}_i, \tag{33c}$$

or

$$E_- = -\partial_i \mathcal{A}_i, \tag{34a}$$

$$B_- = \epsilon_{ij} \partial_i \mathcal{A}_j, \tag{34b}$$

$$B_i = \partial_- \mathcal{A}_i - \partial_i \Delta_\perp^{-1} J^+ + \epsilon_{ij} \partial_j \Delta_\perp^{-1} K^+. \tag{34c}$$

Apparently this arbitrariness is annoying as an ambiguity of our approach, however we interpret it as a manifestation of different possible choices of independent modes for the interacting system. Every different choice is self-consistent and various choices should ultimately lead to the same physical predictions, though they may differ at the intermediate stages. Our novel canonical procedure offers some extra hints for a choice of parametrization. If we demand that the translation generator P^+ retains its free field forms

$$P^+ = \int \mathrm{d}^3\bar{x}\, B_i^2 = \int \mathrm{d}^3\bar{x} (\partial_- \mathcal{A}_i)^2, \tag{35}$$

then the only acceptable parametrization for B_i is (33c). The nonlocal (in x^-) dependence of electromagnetic fields E_- and B_- on J^+ and K^+, given by (33a) and (33b), can be interpreted as constant vector strings. Since these strings appear while we solve the Gauss law constraint equations, we may call them the Gauss law strings. A more detailed discussion of these strings and their relation to the usual Dirac strings is presented in appendix B.

Now it is quite an easy exercise to check that all equations of motion (31a)–(31c) are equivalent to the equation of motion for $\mathcal{A}_i$ fields

$$(2\partial_+\partial_- - \Delta_\perp)\mathcal{A}_i = \epsilon_{ij} \partial_j (\partial_-)^{-1} K^+ + \partial_i (\partial_-)^{-1} J^+ + J^i + \epsilon_{ij} K^j, \tag{36}$$

provided the covariant conservations laws for the electric and magnetic currents (29a), (29b) are taken into account. Thus we see that in our gauge invariant description of Maxwell's theory with electric and magnetic external currents there are still two independent dynamical degrees of freedom, just like in the free field case, though now we have also constant string structure $(\partial_-)^{-1}$ for the interaction terms.

The canonical Poisson brackets for $\mathcal{A}_i$ fields follow immediately from the trivial equation

$$\partial_- \mathcal{A}_i = \{\mathcal{A}_i, P^+\}_{PB}, \tag{37}$$

with P^+ given by (35), leading to

$$2\{\partial_- \mathcal{A}_i(x^+, \bar{x}), \mathcal{A}_j(x^+, \bar{y})\}_{PB} = -\delta_{ij}\delta^3(\bar{x} - \bar{y}), \tag{38}$$

which is precisely the free field result. Similarly, also P^i is a kinematic generator

$$P^i = -\int \mathrm{d}^3\bar{x} (\partial_i \mathcal{A}_j \partial_- \mathcal{A}_j), \tag{39}$$

but P^-, being the LF Hamiltonian, should contain interaction terms. The very form of P^- should be consistent with the effective equation of motion (36), which we write as the Poisson bracket equation

$$2\partial_+\partial_- \mathcal{A}_i = 2\{\partial_- \mathcal{A}_i, P^-\}_{PB} = \Delta_\perp \mathcal{A}_i + \epsilon_{ij} \partial_j (\partial_-)^{-1} K^+ + \partial_i (\partial_-)^{-1} J^+ + J^i + \epsilon_{ij} K^j. \tag{40a}$$

Next, due to (38), we infer the functional differential equation for P^-

$$-\frac{\delta P^-}{\delta \mathcal{A}_i} = \Delta_\perp \mathcal{A}_i + \epsilon_{ij}\partial_j(\partial_-)^{-1}K^+ + \partial_i(\partial_-)^{-1}J^+ + J^i + \epsilon_{ij}K^j, \tag{40b}$$

which can be solved as

$$P^- = \frac{1}{2}\int \mathrm{d}^3\bar{x}[(\partial_i\mathcal{A}_i)^2 + (\epsilon_{ij}\partial_i\mathcal{A}_j)^2] + \int \mathrm{d}^3\bar{x}[\partial_i\mathcal{A}_i(\partial_-)^{-1}J^+ - \epsilon_{ij}\partial_i\mathcal{A}_j(\partial_-)^{-1}K^+] - \int \mathrm{d}^3\bar{x}\mathcal{A}_i(J^i + \epsilon_{ij}K^j) + H_{\mathrm{cur}}. \tag{41}$$

H_{cur} is a constant of functional integration, which may depend on external currents J^μ and K^μ and we will call it the current Hamiltonian.

We note that the above canonical procedure is invariant under the LF electromagnetic duality transformation, which again has the simple form for the independent modes

$$\mathcal{A}_i \longmapsto -\epsilon_{ij}\mathcal{A}_j, \tag{42}$$

provided the current Hamiltonian in (41) is also invariant under the duality transformation of external currents (10*b*). Actually, the determination H_{cur} is not a trivial task and lies beyond the scope of the canonical quantization procedure. In the next section, we will demand a covariant form of the perturbative propagators and this will uniquely fix H_{cur}.

As a concluding remark of this section, we stress that even though the expression for the LF Hamiltonian (41) is quite similar to the well-known LF Hamiltonian for the LC gauge $A_- = 0$, this similarity is quite misleading—one must be aware that there is no way to derive this Hamiltonian within the one gauge potential approach and this explains the failure of [4] for the case of interacting Maxwell's theory.

6. Path integral for electric and magnetic sources

The canonical structure, which we have found in the previous section, forms a good starting point for the canonical quantization procedure for the sector of electromagnetic fields. There is yet an undetermined current Hamiltonian H_{cur}, which describes the LF instantaneous interaction of currents and we have to fix it ultimately. We believe that the simplest way to find the consistent current Hamiltonian H_{cur} follows from the structure of perturbative propagators. This is based on the observation that, within the LF formulation, quite frequently one has to supplement a chronological (in x^+) product of quantum field operators by some LF instantaneous contribution from a Hamiltonian, when one finds a LF perturbative propagator. Since perturbative propagators are derived most easily from a path-integral definition of a generating functional, therefore in this section we will depart from the canonical quantization procedure but rather concentrate on the path-integral formulation.

We define the generating functional of all Green functions as the phase-space path integral, which follows naturally from the canonical structure given in the previous section[4]

$$Z[J^+, J^i, K^+, K^i] = \int \mathcal{D}\mathcal{A}_i \exp\left(\mathrm{i}\int \mathrm{d}^4x\, \partial_+\mathcal{A}_i\partial_-\mathcal{A}_i\right)\exp\left(-\mathrm{i}\int \mathrm{d}x^+ P^-\right). \tag{43}$$

Here we stress that the path-integral field variables $\mathcal{A}_i$ are the gauge invariant quantities and they are the canonical variables in the unconstrained Hamiltonian phase space. Since all

[4] We will omit the normalization constants for all path integrals which will appear in this section keeping in mind the normalization condition $Z[0] = 1$.

integrals are Gaussian we may immediately perform them and get the result

$$Z[J^+, J^i, K^+, K^i] = \exp\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, K^\mu(x)\mathcal{G}_{\mu\nu}(x-y)K^\nu(y)$$
$$\times \exp\frac{\mathrm{i}}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J^\mu(x)\mathcal{G}_{\mu\nu}(x-y)J^\nu(y)$$
$$\times \exp\mathrm{i}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J^\mu(x)\widetilde{\mathcal{D}}_{\mu\nu}(x-y)K^\nu(y)\exp\left(-\mathrm{i}\int \mathrm{d}x^+\, H_{\mathrm{cur}}\right), \tag{44}$$

with the propagators

$$\mathcal{G}_{\mu\nu} = (-g_{\mu\nu} + (n_\mu\partial_\nu + n_\nu\partial_\mu)(n\cdot\partial)^{-1})D_F - n_\mu n_\mu(n\cdot\partial)^{-1}(n\cdot\partial)^{-1}, \tag{45a}$$
$$\widetilde{\mathcal{D}}_{\mu\nu} = \epsilon_{\mu\nu\lambda\rho}n^\lambda\partial^\rho(n\cdot\partial)^{-1}D_F, \tag{45b}$$

where n_μ is the null vector ($n_+ = 1, \bar{n} = 0$), thus $(n\cdot\partial)^{-1} = (\partial_-)^{-1}$. The instantaneous part in (45*a*) can be removed if we choose the current Hamiltonian as

$$H_{\mathrm{cur}} = \frac{1}{2}\int \mathrm{d}^3x[(\partial_-)^{-1}J^+]^2 + \frac{1}{2}\int \mathrm{d}^3x[(\partial_-)^{-1}K^+]^2. \tag{46}$$

This leads to the effective propagators for the electric–electric and magnetic–magnetic sectors

$$\mathcal{D}_{\mu\nu} = (-g_{\mu\nu} + (n_\mu\partial_\nu + n_\nu\partial_\mu)(n\cdot\partial)^{-1})D_F, \tag{47}$$

which have the form of the Abelian gauge field propagator for the LC gauge condition: $n_\mu A^\mu = A^+ = A_-$. Our propagators depend on the null vector n^μ, which enters into the theory due to the LF quantization procedure. However, when the external currents are conserved, then the dependence on n^μ disappears for the propagator $\mathcal{D}_{\mu\nu}$ but remains for the propagator $\widetilde{\mathcal{D}}_{\mu\nu}$. This means that the Lorentz invariance is broken by the quantum interaction of electric and magnetic currents. This is a real challenge to prove that for the physical observables (like cross sections etc) one may restore the Lorentz symmetry. Within the ET formulation, one may prove [18–20] that the dependence on a space like vector n^μ finally disappears. We hope that a similar phenomenon happens also for (47), though here we have a dependence on a null vector $n^2 = 0$.

The Fourier transform of these propagators contains the principal value (PV) prescription for the noncovariant pole $PV\dfrac{1}{k_-}$. It is known, within the ET formulation, that this prescription is not consistent for a non-Abelian gauge field theory [17]. In our gauge invariant procedure, we see that the PV prescription is the only possibility for solving the LF constraint equations in terms of a real valued distribution, at fixed x^+. Thus the status of the PV prescription is consistent here within the LF canonical formulation of Maxwell's theory.

The form of our perturbative propagators strongly suggests that there should be some gauge field model, which effectively produces the same path integral as we have found here. Since there are non-local interaction terms in the current Hamiltonian (46), thus we may add two auxiliary path-integral variables A_+ and C_+ into the phase-space path integral (43). This allows us to write the equivalent expression for the generating functional

$$Z[J^+, J^i, K^+, K^i] = \int \mathcal{D}A_i\mathcal{D}A_+\mathcal{D}C_+ \exp\mathrm{i}\int \mathrm{d}^4x[\mathcal{L}_{\mathrm{local}} + C_+K^+ + A_+J^+ + A_i(J^i + \epsilon_{ij}K^j)], \tag{48a}$$

where the local Lagrangian density $\mathcal{L}_{\mathrm{local}}$ is

$$\mathcal{L}_{\mathrm{local}} = \tfrac{1}{2}(\partial_-A_+)^2 + (\partial_+A_i - \partial_iA_+)\partial_-A_i + \tfrac{1}{2}(\partial_-C_+)^2 + \epsilon_{ij}\partial_iA_j\partial_-C_+. \tag{48b}$$

In these expressions we have changed our gauge invariant path-integral variables $\mathcal{A}_i$ into A_i, since here they are just dummy variables of path integrations.

In the next step we treat this local Lagrangian density $\mathcal{L}_{\rm local}$ as in the case of the double LC gauge condition $A_- = C_- = 0$ imposed on the gauge invariant Lagrangian density

$$\mathcal{L}_{\rm inv} = \tfrac{1}{2}(\partial_+ A_- - \partial_- A_+)^2 + (\partial_+ A_i - \partial_i A_+)(\partial_- A_i - \partial_i A_-) - \tfrac{1}{2}(\epsilon_{ij}\partial_i A_j)^2 + \tfrac{1}{2}(\partial_+ C_- - \partial_- C_+ - \epsilon_{ij}\partial_i A_j)^2, \tag{49}$$

where we have two independent Abelian gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \Theta_e, \tag{50a}$$

$$C_\pm \to C_\pm + \partial_\pm \Theta_m. \tag{50b}$$

The first three terms in (49) form a standard LF Abelian Lagrangian with the gauge potentials A_μ, while the last term describes the contribution of the dual potentials $C_\pm$. We stress that without this last term one cannot consistently quantize Maxwell's theory with both electric and magnetic currents.

Since in previous sections, we kept our canonical procedure explicitly duality invariant, thus we would like to check whether (49) has some duality symmetry. We find that this Lagrangian density is invariant (up to a total derivative) under the following transformation:

$$A_i \longmapsto -\epsilon_{ij} A_j + (\partial_-)^{-1}(\partial_i C_- + \epsilon_{ij}\partial_j A_-), \tag{51a}$$

$$A_\pm \longmapsto C_\pm \longmapsto -A_\pm. \tag{51b}$$

Since for the double LC gauge condition $A_- = C_- = 0$, these transformations reduce to our former gauge invariant duality transformation (42), thus we argue that they are the generalized LF electromagnetic duality transformations for the gauge and dual potentials.

With the transformations (51*a*) and (51*b*), supplemented with the transformation of currents (10*a*), we can check that the duality symmetry (up to a total derivative) is possessed by the interaction Lagrangian density

$$\mathcal{L}_{\rm int} = A_-(J^- - \epsilon_{ij}(\partial_-)^{-1}\partial_i K^j) + C_-(K^- + (\partial_-)^{-1}\partial_i K^i) + A_+ J^+ + C_+ K^+ + A_i(J^i + \epsilon_{ij} K^j). \tag{52}$$

Apparently this interaction Lagrangian is not satisfactory, since the gauge and dual potentials couple nonlocally to external currents. We could introduce further auxiliary variables to keep all potential–current couplings local but at the price of introducing the Lagrange multiplier fields—this would be another manifestation of the canonical constraint (15*c*). However, we have decided not to proceed in this direction here and to discuss it elsewhere.

Though the interaction Lagrangian (52) looks strange, it behaves properly under the gauge transformations. From the transformation (50*a*) one gets the electric current conservations law $\partial_\mu J^\mu = 0$, while from (50*b*) one gets the magnetic current conservation law $\partial_\mu K^\mu = 0$. This convinces us that we may treat the gauge invariant Lagrangian (49) with the interaction Lagrangian (52) as a good starting point for the canonical quantization procedure with different gauge conditions [21].

7. Conclusions and further prospects

In this paper, we show that the electromagnetic duality can be consistently defined for the LF Maxwell theory with the classical external electric and magnetic currents. For practical reasons we introduce the independent physical fields $\mathcal{A}_i$ which allow us to satisfy identically the LF electric and magnetic Gauss law equations. Remembering that fields $\mathcal{A}_i$ are not the transverse components of some gauge potential, we are satisfied that for these physical modes the LF duality transformation looks like the Susskind conjecture $\mathcal{A}_i \to -\epsilon_{ij}\mathcal{A}_j$.

We propose a novel canonical LF procedure, which is based on the longitudinal translation operator P^+, which, being a kinematic generator, has the same form both for free and interacting theories. Usually the canonical procedure starts with a Lagrangian, but in the case of Maxwell's theory with electric and magnetic currents, the starting point is Maxwell's equations. In our novel canonical procedure, we can safely take P^+ from the free field theory, while the LF Hamiltonian P^- can be found as a solution of the functional differential equation.

We also use the path-integral formulation for achieving two different goals. First, we find the perturbative propagators with the instantaneous (in x^+) terms, which then can be cancelled by the contribution from the LF Hamiltonian. The effective propagators (45*b*), (47) have the form known from Schwinger's source theory [16] for an arbitrary constant vector n^μ. Here we present their LF canonical derivation for the case of a null vector $n^2 = 0$. Second, we prove that our gauge invariant formalism is equivalent to some gauge theory with two potentials. We show that the LC gauge condition can be chosen for both the gauge and dual potentials. Our two gauge potential Lagrangians (49) and (52) are far simpler than the one proposed long ago by Zwanziger [6] and recently used for the LF quantization by Mukherjee and Bhattacharya [5]. Here we would like to point out that our result disagrees with [5], where the Hamiltonian contains an additional instantaneous term (for the interaction of electric and magnetic currents), which explicitly breaks the rotational symmetry. In our case the rotational symmetry is broken solely by a choice of the LF surface.

The two gauge potential Lagrangian densities (49) and (52) can form a starting point for the dual formulation of Abelian theory within the LF approach. When further such formulation is generalized to a non-Abelian theory, then a very encouraging possibility arises—the LF version of the dual superconductor models of colour confinement (for an introduction and earlier references see [22]).

When one performs the perturbative calculations, with currents generated by some charged matter fields, then inevitably the perturbative propagators (45*b*), (47) will produce momentum integrals with the UV divergences. Therefore, one will have to regularize the model without spoiling the gauge and duality invariance. Since the antisymmetrical symbol ϵ_{ij} appears explicitly in many expressions, then the dimensional regularization seems to be impractical here and one should look for a kind of the Pauli–Villars regularization. Possibly, this can give rise to the regularized Lagrangian density with higher derivative terms (see [23]).

When the charged matter is treated as a part of the quantum dynamical system, then one may check the second Susskind conjecture [3], that in the LF approach there is no need for any other non-localities, like Dirac strings, beside the usual $(\partial_-)^{-1}$ integral operator. Such investigations should also answer another question: how the Dirac–Schwinger quantization condition for electric and magnetic charges [11, 24] appears within the LF formalism. We hope to give definite answers to these problems in our future publication.

Appendix A. The LF notation

We use the natural units $c = \hbar = 1$. Our LF notation starts with the definitions of null components for the coordinates $x^\pm = (x^0 \pm x^1)/\sqrt{2}$, while the transverse components are $x^i = (x^2, x^3)$. Similar definitions are taken for any 4-vectors. The LF surface coordinates are denoted as $\bar{x} = (x^-, x^i)$. The partial derivatives are taken with respect to contravariant coordinates, thus we have $\partial_+ = \partial/\partial x^+, \partial_- = \partial/\partial x^-, \partial_i = \partial/\partial x^i$. The metric tensor has non-vanishing components $g_{+-} = g_{-+} = 1$, $g_{ij} = -\delta_{ij}$. The scalar product of 4-vectors is $a \cdot b = a_+ b_- + a_- b_- - a_i b_i$, while for the LF surface components we have $\bar{a} \cdot \bar{b} = a^- b_- - a_i b_i$. There are two natural antisymmetric tensors $\epsilon_{ij} = -\epsilon_{ji}$ with $\epsilon_{23} = 1$, and $\epsilon_{+-ij} = \epsilon_{ij}$.

The inverse differential operator $(\partial_-)^{-1}$ is taken as the distribution

$$(\partial_-)^{-1}(\bar{x}) = \tfrac{1}{2}\mathrm{sgn}(x^-)\delta^2(x^i), \tag{A.1}$$

which means that we use the PV prescription in its Fourier integral representation

$$(\partial_-)^{-1}(\bar{x}) = -\mathrm{i}\int \frac{\mathrm{d}^3\bar{k}}{(2\pi)^3}\,\mathrm{e}^{\mathrm{i}\bar{x}\cdot\bar{k}}\mathrm{PV}\frac{1}{k_-}. \tag{A.2}$$

Another integral operator $\partial_i \Delta_\perp^{-1}$ is given as

$$\partial_i \Delta_\perp^{-1}(\bar{x}) = \frac{1}{2\pi}\frac{x^i}{x_\perp^2}\delta(x^-), \qquad x_\perp^2 = x^i x^i, \tag{A.3}$$

while the covariant propagator function is defined as

$$D_F(x) = \int \mathrm{d}^4k \frac{\mathrm{e}^{-\mathrm{i}k\cdot x}}{2k_+k_- - k_\perp^2 + \mathrm{i}\epsilon}. \tag{A.4}$$

Appendix B. Dirac's strings within the ET and LF formulations

Dirac's approach with strings [11, 12] is basically a gauge potential formulation, where the modifications in Bianchi identities (2*b*) are compensated by the redefinition of the electromagnetic field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \tfrac{1}{2}\epsilon_{\mu\nu\lambda\rho}G^{\lambda\rho}, \tag{B.1}$$

provided the string tensor $G^{\lambda\rho}$ satisfies the equation

$$\partial_\lambda G^{\lambda\rho} = K^\rho. \tag{B.2}$$

A straight line string is given by

$$G^{\lambda\rho} = (n\cdot\partial)^{-1}(n^\lambda K^\rho - n^\rho K^\lambda) \tag{B.3}$$

where n^μ is a given fixed vector. In the ET approach one chooses this fixed vector to be space-like $n^\mu = (0, \vec{n})$, thus the integral operator $(n\cdot\partial)^{-1} = -(\vec{n}\cdot\vec{\nabla})^{-1}$ does not include temporal evolution. The orientation of $\vec{n}$ is arbitrary and no physical result should depend on it.

This procedure can be also applied in the LF approach and one finds that there are two different possibilities—one can take either $(n^- = 1, n^+ = n^i = 0)$ or $(n^+ = n^- = 0, n^i \neq 0)$—which lead to no temporal evolution in the integral operator $(n\cdot\partial)^{-1}$. In the first case one has a fixed null vector $n^2 = 0$, while in the second case one again has a space-like vector $n^2 < 0$.

However, we stress that equation (B.2) is not duality invariant since it contains only the magnetic sources K^ρ. Since our present paper is devoted to the gauge and duality invariant formulation we will not discuss such Dirac strings any further here but postpone a more detailed analysis to a future publication.

Within the Hamiltonian formulation, one can adopt Dirac's idea (B.1) for transforming inhomogeneous constraint equations into homogeneous equations. One may define the electromagnetic field tensor as

$$F_{\mu\nu} = \mathcal{F}_{\mu\nu} + n_\mu f_\nu - n_\nu f_\mu + \epsilon_{\mu\nu\lambda\rho}n^\lambda g^\rho, \tag{B.4}$$

where the components of $\mathcal{F}_{\mu\nu}$ satisfy the homogeneous constraint equations, while f^μ and g^μ are some functions which may depend on electric and magnetic charges. This modification is duality invariant, provided one introduces the duality transformation for f^μ and g^μ

$$g^\mu \mapsto f^\mu \mapsto -g^\mu. \tag{B.5}$$

The fixed vector n^μ is quite arbitrary and here we will discuss only the simplest choices. In the ET approach one can take $(n^0 = 1, \vec{n} = 0)$ which leads to

$$\vec{E} = \vec{\mathcal{E}} + \vec{f}, \qquad \vec{\nabla} \cdot \vec{f} = J^0, \tag{B.6}$$

$$\vec{B} = \vec{\mathcal{B}} - \vec{g}, \qquad \vec{\nabla} \cdot \vec{g} = -K^0. \tag{B.7}$$

In the LF approach one can take $\left(n^- = 1, n^+ = n^i = 0\right)$ which leads to

$$E_- = \mathcal{E}_- - f_+, \qquad \partial_- f_+ = J^+, \tag{B.8}$$

$$B_- = \mathcal{B}_- + g_+, \qquad \partial_- g_+ = -K^+, \tag{B.9}$$

$$E_i = \mathcal{E}_i, \qquad B_i = \mathcal{B}_i. \tag{B.10}$$

These two choices of a constant vector n^μ are equivalent, since they indicate the respective temporal evolution parameters: in the ET case we have $n_\mu x^\mu = x^0$, while in the LF case we have $n_\mu x^\mu = x^+$. Evidently equations (B.8)–(B.10) are the solutions (33*a*)–(33*c*) proposed in the main text, while equations (B.6)–(B.7) lead to Coulomb-like solutions for electric and magnetic monopoles.

References

[1] Gambini R and Salamó S 1979 *Nuovo Cimento* A **49** 276
[2] Cabibbo N and Ferrari E 1966 *Nuovo Cimento* **23** 1147
[3] Susskind L 1996 Invited talk at the *Int. Workshop on Light-Front Quantization and Non-Perturbative QCD (Ames, IA, 3–14 June 1996)* (unpublished)
[4] Brisudova M M 1999 *Phys. Rev.* D **59** 087702
[5] Mukherjee A and Bhattacharya S 2000 *Int. J. Mod. Phys.* A **15** 4739
[6] Zwanziger D 1971 *Phys. Rev.* D **3** 880
[7] Dirac P A M 1958 *Proc. R. Soc.* A **246** 326
[8] Faddeev L and Jackiw R 1988 *Phys. Rev. Lett.* **60** 1692
[9] Brodsky S J, McCartor G, Pauli H-C and Pinsky S S 1993 *Part. World* **3** 109
[10] Brodsky S J, Pauli H-C and Pinsky S S 1998 *Phys. Rep.* **301** 299
[11] Dirac P A M 1931 *Proc. R. Soc.* A **133** 60
[12] Dirac P A M 1948 *Phys. Rev.* **74** 817
[13] Wu T T and Yang C N 1975 *Phys. Rev.* D **12** 3845
[14] Białynicki-Birula I and Białynicka-Birula Z 1975 *Quantum Electrodynamics* (Oxford: Pergamon)
[15] Przeszowski J A 2003 *Proc. X Mexican School of Fields and Particles* (New York: AIP) vol 670 p 188
[16] Schwinger J 1968 *Phys. Rev.* **173** 1536
Schwinger J 1975 *Phys. Rev.* D **12** 3105
[17] Bassetto A, Nardelli G and Soldati R 1991 *Yang–Mills Theories In Algebraic Noncovariant Gauges: Canonical Quantization and Renormalization* (Singapore: World Scientific)
[18] Urrutia L F 1978 *Phys. Rev.* D **18** 3031
[19] Gamberg L and Milton K A 2000 *Phys. Rev.* D **61** 075013
[20] Milton K A, Kalbfleish G R, Lou W and Gamberg L 2002 *Int. J. Mod. Phys.* A **17** 732
[21] Dzimida E 2005 *Few-Body Systems* **36** 87
[22] Ripka G 2004 *Dual Superconductor Models of Color Confinement (Lecture Notes in Physics)* (Berlin: Springer)
[23] Panagiotakoulos C 1982 *Nucl. Phys.* **198** 303
[24] Schwinger J 1966 *Phys. Rev.* **144** 1087